ON CONTACT OF CURVES AND SURFACES.

by

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Introduction.

There are curves and surfaces concerning which we know very little except as we study them in relationship with known curves or surfaces. If an unknown curve C' merely crosses some curve C we can gain a little information as to the nature of C' in the neighborhood of that point. If the curves have a tangent in common we know still more about our curve C'. The more similar the curves at a point the more we can learn about C' by studying the properties of C in the neighborhood of the common point. The above is true concerning surfaces also, or concerning curves and surfaces. The theory of contact has been developed in order that we might thus study the unknown by means of some known curve or surface.

Picard* has given a general treatment of this subject and Goursat* has given a treatment for plane curves that might be extended for space curves and surfaces. This discussion, however, follows that general theory of Picard, but listing in addition, examples to illustrate the methods suggested by it.

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*Picard: Traite D'Analyse. Vol. 1.

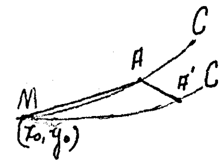
*Goursat-Hedrick: Mathematical Analysis. Vol. 1.

Some Preliminary Facts Concerning Infinitesimals.

A variable which decreases indefinitely, i. e., a variable which approaches the limit zero, is called an infinitesimal. Whenever we have occasion to consider several infinitesimals connected by some law, we arbitrarily choose some one as the principal infinitesimal. Any infinitesimal such that the limit of its ratio to the principal infinitesimal is finite is called an infinitesimal of the first order. An infinitesimal such that the limit of its ratio to the square of the principal infinitesimal is finite is called an infinitesimal of the second order. In the same way, an infinitesimal such that the limit of its ratio to the $\underline{n}$ th power of the principal infinitesimal is finite is called an infinitesimal of the $\underline{n}$ th order.

When dealing with infinitesimals that arise in connection with a study on the contact of curves, one of the first questions to be answered is the relationship that exists between the order of a chord and the order of its arc. However, the definition of the length of arc of a curve as the limiting position of the sum of the chords requires that the ratio of the chord to the arc approach unity. This makes the chord and its arc infinitesimals of the same order.

Contact of Plane Curves.



Let C and C' be two plane curves having a point $M(x_0, y_0)$ in common. Let the equations of the curves be represented as follows:

$$C \begin{cases} x = f(t), \\ y = F(t), \end{cases} \qquad C' \begin{cases} x = \varphi(u), \\ y = \Phi(u). \end{cases}$$

Let the point $M(x_0, y_0)$ correspond to $t = t_0$ and suppose $f'(t_0)$, $F'(t_0)$ not both equal to zero. On C', let $x_0 = \varphi(u_0)$ and $y_0 = \Phi(u_0)$. Suppose further that $\varphi'(u_0)$, $\Phi'(u_0)$ do not both equal zero. To every point A on C let us establish a corresponding point A' on C'. Using Taylor's formula and considering $u - u_0$ as a function of $t - t_0$, we have the following expression:

$$u - u_0 = \lambda_1 (t - t_0) + \lambda_2 (t - t_0)^2 + \cdots\cdots + \lambda_n (t - t_0)^n + \cdots\cdots .$$

Considering our curve C, we have

$$\overline{MA}^2 = [f(t) - f(t_0)]^2 + [F(t) - F(t_0)]^2 .$$

But by Taylor's expansion

$$f(t) = f(t_0) + (t - t_0) f'(t_0) + \frac{(t - t_0)^2}{\underline{|2}} f''(t_0) + \cdots\cdots ,$$

and

$$F(t) = F(t_0) + (t - t_0) F'(t_0) + \frac{(t - t_0)^2}{\underline{|2}} F''(t_0) + \cdots\cdots .$$

Hence

$$\overline{MA}^2 = (t - t_0)^2 \left[f'^2(t_0) + F'^2(t_0)\right] + (t - t_0)^3 \left[f'(t_0) f''(t_0) + F'(t_0) F''(t_0)\right] + \cdots ,$$

or

$$\frac{\overline{MA}^2}{(t - t_0)^2} = f'^2(t_0) + F'^2(t_0) + (t - t_0)\left[f'(t_0) f''(t_0) + F'(t_0) F''(t_0)\right] + \cdots\cdots .$$

But since

$$f'^2(t_0) + F'^2(t_0) \neq 0$$

MA is an infinitesimal of the same order as $t - t_0$. Thus the

$\widehat{MA}$, the chord MA and $t-t_0$ are infinitesimals of the same order and any one may be used as the principal infinitesimal without changing the results as to order. In the following discussion let us use $t-t_0$ as our principal infinitesimal and investigate the conditions necessary for the distance AA' to be an infinitesimal of the order n+1 with respect to $t-t_0$.

Concerning the infinitesimal AA', we have

$$\overline{AA'}^2 = (x'-x)^2 + (y'-y)^2 .$$

Consequently in order that AA' be of order n+1 with respect to $t-t_0$, it is necessary that both $x'-x$ and $y'-y$ be infinitesimals of order n+1 with respect to $t-t_0$.

Using Taylor's expansion, we have

$$x = f(t_0) + (t-t_0) f'(t_0) + \frac{(t-t_0)^2}{\underline{|2}} f''(t_0) + \cdots\cdots,$$

and

$$x' = \phi(u_0) + (u-u_0)\phi'(u_0) + \frac{(u-u_0)^2}{\underline{|2}}\phi''(u_0) + \cdots\cdots,$$

or

$$x' = \phi(u_0) + \lambda_1(t-t_0)\phi'(u_0) + \lambda_2(t-t_0)^2\phi'(u_0) + \frac{\lambda_1^2(t-t_0)^2}{\underline{|2}}\phi''(u_0) + \cdots,$$

But

$$f(t_0) = \phi(u_0) .$$

So for $x'-x$ to be of order n+1 with respect to $t-t_0$ the following n equations in n unknowns must be true:

$$\lambda_1\phi'(u_0) - f'(t_0) = 0,$$

$$\frac{\lambda_1^2}{\underline{|2}}\phi''(u_0) + \lambda_2\phi'(u_0) - \frac{f''(t_0)}{\underline{|2}} = 0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

The difference $y'-y$ may now be expressed in like manner by replacing f and ϕ by F and Φ. This gives the n necessary and sufficient conditions concerning the order of $y'-y$, namely that:

$$\lambda_1\Phi'(u_0) - F'(t_0) = 0,$$

$$\frac{\lambda_1^2\Phi''(u_0)}{\underline{|2}} + \lambda_2\Phi'(u_0) - F''(t_0) = 0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

We now have <u>2n</u> equations to determine the <u>n</u> unknowns $\lambda_1, \lambda_2, \cdots \lambda_n$. Since $f'(t_0)$ and $F'(t_0)$ cannot both vanish at once nor can $\phi'(u_0)$ and $\Phi'(u_0)$ both vanish at once, we can always obtain a value of λ_1 from the first equation in one of the sets. By substituting this value in the next equation we can obtain the value of λ_2. In like manner, the values of the memaining unknowns can be found. Then, by substituting their values in the other set of equations we have the <u>n</u> <u>necessary and sufficient conditions that our infinitesimal be of order n+1 with respect to $t-t_0$</u>. If the infinitesimal is of order <u>n+1</u>, our curves are said to have contact of order <u>n</u>. The higher the order of contact, the more closely the curves correspond.

Furthermore, if $x'-x$ and $y'-y$ are both infinitesimals of order <u>n+1</u>, we have:

$$x'-x = \frac{(t-t_0)^{n+1}}{\lfloor n+1}\left[\lambda_1^{n+1}\phi^{(n+1)}(u_0) + \cdots\cdots\right.,$$

$$y'-y = \frac{(t-t_0)^{n+1}}{\lfloor n+1}\left[\lambda_1^{n+1}\phi^{(n+1)}(u_0) + \cdots\cdots\right.$$

If we place $x'=x$, $y'=y$ in order to obtain the points of intersection of the curves, our equations become

$$0 = \frac{(t-t_0)^{n+1}}{\lfloor n+1}\left[\lambda_1^{n+1}\phi^{(n+1)}(u_0) + \cdots\cdots\right.,$$

$$0 = \frac{(t-t_0)^{n+1}}{\lfloor n+1}\left[\lambda_1^{n+1}\Phi^{(n+1)}(u_0) + \cdots\cdots\right.$$

Every value of t which satisfied both of these equations gives us a point common to the two curves. But from the common coefficient, we see that t_0 satisfies them for <u>n+1</u> points. Thus we have <u>n+1</u> coincident points for $t=t_0$.

Since we are assuming our functions to be continuous in the neighborhood of $t=t_0$, the expressions inside the brackets approach the value of the terms not containing $t-t_0$. Consequently, the expressions within the brackets do not appraoch zero as t approaches t_0; and therefore do not change sign as t passes through the value t_0. But, $t-t_0$ does change sign as t passes through the value t_0; hence if n+1 is an odd power the signs of our infinitesimals change, but if n+1 is an even power, the signs remain the same. If the infinitesimals change sign, the curves must cross. Thus, when the order of contact of the curves is odd, they have an even number of coincident point, and they do not cross, but if the order of contact is even they have an odd number of coincident points and the curves do cross.

The first case considered deals with the most general form of our curves. Let us now change our parametric form for the curve C' such that $\phi(u)$ becomes $f(u)$, then the curves have as their equations,

$$C\begin{cases} x=f(t), \\ y=F(t), \end{cases} \qquad C'\begin{cases} x=f(u), \\ y=\Phi(u). \end{cases}$$

Suppose, these curves have some point in common determined by $t_0=u_0$; and suppose also that $f'(t_0)\neq 0$. The conditions of contact are now simplified. Using the same method as before,

$$\lambda_1=1, \quad \lambda_2=\lambda_3= - - - - - - - - - =\lambda_n=0.$$

The equations, of the conditions for contact of order n, then become

$$F'(t_0)=\Phi'(u_0),\ F''(t_0)=\Phi''(u_0),\ - - - - - ,\ F^{(n)}(t_0)=\Phi^{(n)}(u_0).$$

Thus, for the two curves to have contact of order n, it is necessary and sufficient that the functions F and Φ and their first n derivatives be equal for $t=u,\ t_0=u_0$.

In the simple case of two curves being defined by the two equations

$$y=F(x)\quad,\quad y=\Phi(x),$$

the x of each curve may be thought of as being represented by the same function in x, as x=x. Then applying the results of the preceding case, the conditions for contact of order n reduce to

$$F(x_0)=\Phi(x_0),\ F'(x_0)=\Phi'(x_0),\ ---,\ F^{(n)}(x_0)=\Phi^{(n)}(x_0).$$

Finally, our curves might be defined by the relations

$$C\begin{cases}x=f(t),\\ y=F(t),\end{cases}\qquad C'\{\lambda(x,y)=0,$$

with the common point (x_0, y_0) corresponding to $t=t_0$. Replacing x of C' by x of C, we have $\lambda(f(t),y)=0$ defining a certain implicit function, $y=\Phi(t)$, which reduces to y_0 for $t=t_0$. Then, $\lambda(x,y)=0$ may be replaced by

$$x=f(t),\quad y=\Phi(t).$$

Thus, we have the conditions for contact of order n to be as before; namely, that the corresponding first n derivatives for $F(t)$ and $\Phi(t)$ must be equal. However, if

$$\Phi(t_0)=F(t_0),\ \Phi'(t_0)=F'(t_0),\ ----,\ \Phi^{(n)}(t_0)=F^{(n)}(t_0),$$

the derivatives up to ~~the~~ order n+1 are the same for the two functions

$$\lambda[f(t),F(t)]\ \text{and}\ \lambda[f(t),\Phi(t)].$$

But when $\Phi(t)$ is substituted in $\lambda[f(t), y]=0$ the function $\lambda[f(t), \Phi(t)]$ becomes identically zero, and so its derivatives are zero. Therefore, $\lambda[f(t), F(t)]=0$ and its n derivatives must equal zero, if the curves have contact of order n.

Contact of Space Curves.

The theory of contact between space curves is similar to that of plane curves. Let C and C' be two space curves having a common point M. As before, let us take a point A on C an infinitesimal distance from M, and on C' the corresponding point A' an infinitesimal distance from M, and then investigate the conditions which must be satisfied in order for AA' to be of order n+1 with respect to the chord or arc MA.

Considering first the most general case, let us suppose the two curves are represented by the equations:

$$C\begin{cases} x = f(t), \\ y = \varphi(t), \\ z = \psi(t), \end{cases} \qquad C'\begin{cases} x = F(u), \\ y = \Phi(u), \\ z = \Psi(u), \end{cases}$$

and that the common point M corresponds to $t = t_0$, $u = u_0$. Suppose further that the three derivatives f'; φ'; ψ'; are not all equal to zero for $t = t_0$, and likewise that F', Φ', Ψ' are not all equal to zero for $u = u_0$.

In order to establish a correspondence between the two curves let

$$u - u_0 = \lambda_1(t-t_0) + \lambda_2(t-t_0)^2 + \cdots + \lambda_n(t-t_0)^n + \cdots.$$

This defines the relation between u and t and fixes the correspondence between A and A'. We must now determine the λ coefficients such that $x'-x$, $y'-y$ and $z'-z$ will be infinitesimals of order n+1 with respect to $t-t_0$. Following the same plan as for plane curves, we have three groups of n equations each, the first group being;

$$\lambda_1 F'(u_0) - f'(t_0) = 0,$$
$$\lambda_2 F'(u_0) + \frac{\lambda_1^2 F''(u_0)}{\underline{|2}} - \frac{f''(t_0)}{\underline{|2}} = 0,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

The two other groups may be obtained by replacing F and f by Φ and φ, and by Ψ and ψ.

Since the three derivatives, $F'(u_0)$, $\Phi'(u_0)$, $\Psi'(u_0)$ cannot all become zero at once, it is possible to obtain the values of $\lambda_1, \lambda_2, ---, \lambda_n$ from one of the groups. By substituting the values thus obtained in the other two groups we obtain <u>2n equations representing the necessary and sufficient conditions for the infinitesimal to be of order n+1 and for our curves to have contact of order n at M.</u>

Many times our equations are represented in a more simple form. Suppose, for example, that the <u>F</u> function coincides with the <u>f</u> function and that $t_0 = u_0$. The equations of the two curves become,

$$C \begin{cases} x = f(t), \\ y = \varphi(t), \\ z = \psi(t), \end{cases} \qquad C' \begin{cases} x = f(u), \\ y = \Phi(u), \\ z = \Psi(u). \end{cases}$$

Then, $\varphi(t_0) = \Phi(u_0)$ and $\psi(t_0) = \Psi(u_0)$.

The first group of equations give as a result:

$$\lambda_1 = 1, \ \lambda_2 = \lambda_3 = ------ = \lambda_n = 0.$$

Consequently

$$u - u_0 = t - t_0 + \lambda_n (t - t_0)^{n+1} + ----.$$

Using Taylor's expansion:

$$y = \varphi(t_0) + (t - t_0)\varphi'(t_0) + \frac{(t-t_0)^2}{\underline{|2}}\varphi''(t_0) + ---,$$

and

$$y' = \Phi(u_0) + (u - u_0)\Phi'(u_0) + \frac{(u-u_0)^2}{\underline{|2}}\Phi''(u_0) + ----.$$

But, up to and including the <u>n</u> th derivative, $u - u_0 = t - t_0$. <u>Thus, the necessary and sufficient condition, that $y' - y$ be an infinitesimal of order n+1 with respect to $t - t_0$ is that</u>

<u>the first n derivatives for $\phi(t_0)$ and $\Phi(u_0)$ be equal; and similarly for $\psi(t_0)$ and $\Psi(u_0)$ in order that $z'-z$ be of order n+1. The curves then have contact of order n.</u>

Suppose, as a final consideration in space curves, that the curves are defined by the following equations:

$$C\begin{cases} x=f(t), \\ y=\phi(t), \\ z=\psi(t), \end{cases} \qquad C'\begin{cases} F_1(x,y,z)=0, \\ F_2(x,y,z)=0. \end{cases}$$

The conditions for contact of order <u>n</u> at a common point (x_0, y_0, z_0), corresponding to $t=t_0$, may be obtained in practically the same manner as the last case considered in plane curves.

Replace <u>x</u> in <u>F_1</u> by $f(t)$; then $F_1(f(t),y,z)=0$. Likewise replacing <u>x</u> in <u>F_2</u> by $f(t)$, $F_2(f(t),y,z)=0$. Solving these two equations for <u>y</u> and <u>z</u> in terms of <u>t</u>, we obtain $y=\Phi(t)$, $z=\Psi(t)$. Then our curves are given by,

$$C\begin{cases} x=f(t), \\ y=\phi(t), \\ z=\psi(t), \end{cases} \qquad C'\begin{cases} x=f(t), \\ y=\Phi(t), \\ z=\Psi(t). \end{cases}$$

and to have contact of order <u>n</u>, the first <u>n</u> derivatives for $\Phi(t)$ and $\phi(t)$ must be equal, and the first <u>n</u> derivatives for $\Psi(t)$ and $\psi(t)$ must be equal.

Thus, $F_1(f(t),\Phi(t),\Psi(t))$ and $F_1(f(t),\phi(t),\psi(t))$ are equal and their respective derivatives up to order <u>n+1</u> are equal. But $F_1(f(t),\Phi(t),\Psi(t))\equiv 0$ so $F_1(f(t),\phi(t),\psi(t))=0$ and the first <u>n</u> derivatives of $F_1(f(t),\phi(t),\psi(t))$ are zero. Likewise $F_2(f(t),\Phi(t),\Psi(t))$ and $F_2(f(t),\phi(t),\psi(t))$ are equal and their respective derivatives up to order <u>n+1</u> are equal. Following the same reasoning $F_2(f(t),\phi(t),\psi(t))=0$ and the first <u>n</u> derivatives of $F_2(f(t),\phi(t),\psi(t))$ are zero.

Our curves will then have contact of order <u>n</u>, <u>if the functions for x, y, z, of C when substituted in equations for C' make these functions and their first n derivatives equal to zero at the common point.</u>

Contact of a Curve and Surface.

Let us now consider the conditions necessary in order to have contact of order n between a curve C and a surface S having a common Point M. Taking a point A on C, an infinitely small distance from M and a corresponding point A' on S, an infinitely small distance from M, let us determine the conditions which must be satisfied in order that AA' be of order n+1 with respect to MA.

Let the curve C and the surface S be defined by the equations:

$$C\begin{cases} x = f(t), \\ y = \varphi(t), \\ z = \psi(t), \end{cases} \qquad S\begin{cases} x = F(u,v), \\ y = \Phi(u,v), \\ z = \Psi(u,v). \end{cases}$$

Let $t = t_0$, $u = u_0$, $v = v_0$ be the values of the parameters giving the common point M. Suppose that $f'(t_0), \varphi'(t_0), \psi'(t_0)$ do not all equal zero at once, and also that the three functional determinants

$$\frac{D(F,\Phi)}{D(u,v)}, \quad \frac{D(\Phi,\Psi)}{D(u,v)}, \quad \frac{D(\Psi,F)}{D(u,v)},$$

do not annul themselves at the same time for $u = u_0$, $v = v_0$.

In order to establish a correspondence between A and A', let us define u and v as functions of t and let the development be represented as follows:

$$u - u_0 = \lambda_1(t-t_0) + \lambda_2(t-t_0)^2 + \lambda_3(t-t_0)^3 + \cdots,$$

$$v - v_0 = \mu_1(t-t_0) + \mu_2(t-t_0)^2 + \mu_3(t-t_0)^3 + \cdots.$$

Considering the conditions under which the difference $x' - x$

will be of order <u>n+1</u> with respect to $t-t_0$, we have,

$$x = f(t_0) + (t-t_0)f'(t_0) + \frac{(t-t_0)^2}{\lfloor 2}f''(t_0) + \cdots\cdots$$

and

$$x' = F(u_0,v_0) + \left[(u-u_0)\frac{\partial F}{\partial u_0} + (v-v_0)\frac{\partial F}{\partial v_0}\right] + \frac{1}{\lfloor 2}\left[(u-u_0)^2\frac{\partial^2 F}{\partial u_0^2} + 2(u-u_0)(v-v_0)\frac{\partial^2 F}{\partial u_0 \partial v_0} + (v-v_0)^2\frac{\partial^2 F}{\partial v_0^2}\right] + \cdots,$$

or

$$x' = F(u_0,v_0) + \lambda_1(t-t_0)\frac{\partial F}{\partial u_0} + \mu_1(t-t_0)\frac{\partial F}{\partial v_0} + \lambda_2(t-t_0)^2\frac{\partial F}{\partial u_0} + \mu_2(t-t_0)^2\frac{\partial F}{\partial v_0} + \frac{1}{\lfloor 2}\left[\lambda_1^2(t-t_0)^2\frac{\partial^2 F}{\partial u_0^2} + 2\lambda_1\mu_1(t-t_0)^2\frac{\partial^2 F}{\partial u_0 \partial v_0} + \mu_1^2(t-t_0)^2\frac{\partial^2 F}{\partial v_0^2}\right] + \cdots$$

Consequently,

$$\lambda_1 \frac{\partial F}{\partial u_0} + \mu_1 \frac{\partial F}{\partial v_0} - f'(t_0) = 0,$$

$$\lambda_2 \frac{\partial F}{\partial u_0} + \mu_2 \frac{\partial F}{\partial v_0} + \frac{1}{\lfloor 2}\left[\lambda_1^2\frac{\partial^2 F}{\partial u_0^2} + 2\lambda_1\mu_1\frac{\partial^2 F}{\partial u_0 \partial v_0} + \mu_1^2\frac{\partial^2 F}{\partial v_0^2}\right] - \frac{f''(t_0)}{\lfloor 2} = 0,$$

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In the same manner we can obtain <u>n</u> equations which must be satisfied if $y'-y$ be of order <u>n+1</u> and <u>n</u> equations which must be satisfied if $z'-z$ be of order <u>n+1</u>. Thus we have <u>3n</u> equations in which we can solve for the λ's and μ's. Suppose the functional determinant $\frac{D(F,\Phi)}{D(u,v)} \neq 0$ for $u = u_0$; $v = v_0$. Then the values of λ_1 and μ_1 can be obtained by solving the first equations of groups one and two; λ_2 and μ_2 by solving the second equations; and likewise for the others. By substituting these values in the remaining group we obtain the <u>n necessary and sufficient conditions for C and S to have contact of order n at the common point M</u>.

The results are greatly simplified by taking the curve and surface as defined by the equations

$$C\begin{cases} x = f(t), \\ y = \varphi(t), \\ z = \psi(t), \end{cases} \qquad S\begin{cases} x = f(u), \\ y = \varphi(v), \\ z = \psi(u,v). \end{cases}$$

The common point corresponds to the value $t_0 = u_0 = v_0$ of the parameters. The first two groups of equations give

$$\lambda_1 = 1, \quad \lambda_2 = \lambda_3 = \cdots = \lambda_n = 0,$$

$$\mu_1 = 1, \quad \mu_2 = \mu_3 = \cdots = \mu_n = 0.$$

Substituting these values in the third group, for $z' - z$, we have

$$\frac{\partial \Psi}{\partial u_0} + \frac{\partial \Psi}{\partial v_0} - \psi'(t_0) = 0,$$

$$\frac{\partial^2 \Psi}{\partial u_0^2} + 2\frac{\partial^2 \Psi}{\partial u_0 \partial v_0} + \frac{\partial^2 \Psi}{\partial v_0^2} - \psi''(t_0) = 0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

But $\frac{\partial \Psi}{\partial u_0} + \frac{\partial \Psi}{\partial v_0}$ gives the same result as when t is substituted for u and v, the derivative taken with respect to t, and t put equal to t_0. The second equation is the same as $\Psi''(t_0, t_0) - \psi''(t_0) = 0$. Likewise for the remaining equations. In other words, the first n derivatives of $\Psi(t, t)$ and $\psi(t)$ must be equal.

If our equations are given as follows:

$$C\begin{cases} x = f(t), \\ y = \phi(t), \\ z = \psi(t), \end{cases} \qquad S\{\, F(x, y, z) = 0,$$

we can substitute $x = f(u)$ and $y = \phi(v)$ in $F(x, y, z) = 0$. This gives $F(f(u), \phi(v), z) = 0$, whence $z = \Psi(u, v)$. Then we have as our equations

$$C\begin{cases} x = f(t), \\ y = \phi(t), \\ z = \psi(t), \end{cases} \qquad S\begin{cases} x = f(u), \\ y = \phi(v), \\ z = \Psi(u, v). \end{cases}$$

Let the common point correspond to $t_0 = u_0 = v_0$. Then to have contact of order n, we must have

$$\Psi'(t, t) = \psi'(t), \quad \cdots\cdots\cdots, \quad \Psi^{(n)}(t, t) = \psi^{(n)}(t).$$

When this exists $F(f(t), \phi(t), \psi(t))$ and $F(f(u), \phi(v), \Psi(u, v))$ are equal and their respective derivatives up to order n+1 are equal. But $F(f(u), \phi(v), \Psi(u, v)) \equiv 0$, hence the derivatives of $F(f(u), \phi(v), \Psi(u, v))$

equal zero, $F(f(t), \varphi(t), \psi(t))$ equals zero, and its first n derivatives equal zero. Therefore, the necessary and sufficient conditions for contact of order n is that $F(f(t), \varphi(t), \psi(t)) = 0$ and the first n derivatives of $F(f(t), \varphi(t), \psi(t))$ each equals zero.

Contact of Two Surfaces.

For consideration of the most general case of contact between two surfaces, let us represent the equations of the surfaces by

$$S\begin{cases} x=f(u,v), \\ y=\varphi(u,v), \\ z=\psi(u,v), \end{cases} \qquad S'\begin{cases} x=F(U,V), \\ y=\Phi(U,V), \\ z=\Psi(U,V), \end{cases}$$

the common point $M(x_0, y_0 z_0)$ corresponding to (u_0, v_0) and (U_0, V_0).

Let us take a point A on S and a corresponding point A' on S', each an infinitely small distance from M, and determine the conditions under which AA' is an infinitesimal of order <u>n+1</u> with respect to $u-u_0$ and $v-v_0$. Let the correspondence between A and A' be defined as follows:

$$U-U_0 = \lambda_1(u-u_0)+\mu_1(v-v_0)+o_1(u-u_0)^2+\alpha_1(u-u_0)(v-v_0)+\beta_1(v-v_0)^2+\cdots,$$
$$V-V_0 = \lambda_2(u-u_0)+\mu_2(v-v_0)+o_2(u-u_0)^2+\alpha_2(u-u_0)(v-v_0)+\beta_2(v-v_0)^2+\cdots.$$

Using the same plan as before and expanding by Taylor's formula, we have,

$$x' = F(U_0,V_0)+(U-U_0)\frac{\partial F}{\partial U_0}+(V-V_0)\frac{\partial F}{\partial V_0}+\frac{1}{\underline{|2}}\left[(U-U_0)^2\frac{\partial^2 F}{\partial U_0^2}+2(U-U_0)(V-V_0)\frac{\partial^2 F}{\partial U_0\partial V_0}+(V-V_0)^2\frac{\partial^2 F}{\partial V_0^2}\right]+\cdots,$$

or

$$\begin{aligned} x' = {} & F(U_0,V_0)+\left[\lambda_1(u-u_0)+\mu_1(v-v_0)\right]\frac{\partial F}{\partial U_0}+\left[\lambda_2(u-u_0)+\mu_2(v-v_0)\right]\frac{\partial F}{\partial V_0}+\left[o_1(u-u_0)^2+\alpha_1(u-u_0)(v-v_0)+\beta_1(v-v_0)^2\right]\frac{\partial F}{\partial U_0} \\ & +\left[o_2(u-u_0)^2+\alpha_2(u-u_0)(v-v_0)+\beta_2(v-v_0)^2\right]\frac{\partial F}{\partial V_0}+\frac{1}{\underline{|2}}\Big[\left(\lambda_1^2(u-u_0)^2+2\mu_1\lambda_1(u-u_0)(v-v_0)+\mu_1^2(v-v_0)^2\right)\frac{\partial^2 F}{\partial U_0^2} \\ & +2\left(\lambda_1\lambda_2(u-u_0)^2+\lambda_2\mu_1(u-u_0)(v-v_0)+\lambda_1\mu_2(u-u_0)(v-v_0)+\mu_1\mu_2(v-v_0)^2\right)\frac{\partial^2 F}{\partial U_0\partial V_0} \\ & +\left(\lambda_2^2(u-u_0)^2+2\lambda_2\mu_2(u-u_0)(v-v_0)+\mu_2^2(v-v_0)^2\right)\frac{\partial^2 F}{\partial V_0^2}\Big]+\cdots\cdots. \end{aligned}$$

Also

$$x = f(u_0,v_0)+(u-u_0)\frac{\partial f}{\partial u_0}+(v-v_0)\frac{\partial f}{\partial v_0}+\frac{1}{\underline{|2}}\left[(u-u_0)^2\frac{\partial^2 f}{\partial u_0^2}+2(u-u_0)(v-v_0)\frac{\partial^2 f}{\partial u_0\partial v_0}+(v-v_0)^2\frac{\partial^2 f}{\partial v_0^2}\right]+\cdots.$$

So, for $x'-x$ to be an infinitesimal of order <u>n+1</u>, we must have

$$\lambda_1\frac{\partial F}{\partial U_0}+\lambda_2\frac{\partial F}{\partial V_0}-\frac{\partial f}{\partial u_0}=0,$$
$$\mu_1\frac{\partial F}{\partial U_0}+\mu_2\frac{\partial F}{\partial V_0}-\frac{\partial f}{\partial v_0}=0,$$

$$a_1\frac{\partial F}{\partial U_0}+a_2\frac{\partial F}{\partial V_0}+\frac{\lambda_1^2}{\lfloor 2}\frac{\partial^2 F}{\partial U_0^2}+\lambda_1\lambda_2\frac{\partial^2 F}{\partial U_0\partial V_0}+\frac{\lambda_2^2}{\lfloor 2}\frac{\partial^2 F}{\partial V_0^2}-\frac{1}{\lfloor 2}\frac{\partial^2 f}{\partial u_0^2}=0,$$
$$\alpha_1\frac{\partial F}{\partial U_0}+\alpha_2\frac{\partial F}{\partial V_0}+\mu_1\lambda_1\frac{\partial^2 F}{\partial U_0^2}+\lambda_2\mu_1\frac{\partial^2 F}{\partial U_0\partial V_0}+\lambda_1\mu_2\frac{\partial^2 F}{\partial U_0\partial V_0}+\lambda_2\mu_2\frac{\partial^2 F}{\partial V_0^2}-\frac{\partial^2 f}{\partial u_0\partial v_0}=0,$$
$$\beta_1\frac{\partial F}{\partial U_0}+\beta_2\frac{\partial F}{\partial V_0}+\frac{\mu_1^2}{\lfloor 2}\frac{\partial^2 F}{\partial U_0^2}+\mu_1\mu_2\frac{\partial^2 F}{\partial U_0\partial V_0}+\frac{\mu_2^2}{\lfloor 2}\frac{\partial^2 F}{\partial V_0^2}-\frac{1}{\lfloor 2}\frac{\partial^2 f}{\partial v_0^2}=0,$$
$$\cdots\cdots\cdots\cdots$$

It is necessary for all powers of $u-u_0$ and $v-v_0$ up to and including the <u>n</u>th to be eliminated. The equations, however, form an arithmetical progression and the sum of the equations for <u>n</u> powers is

$$\frac{n}{2}[n+3].$$

We have the same number for $y'-y$ and for $z'-z$. Hence, the total number of equations is

$$3\left[\frac{n}{2}(n+3)\right].$$

From $y'-y$ we have

$$\lambda_1\frac{\partial\Phi}{\partial U_0}+\lambda_2\frac{\partial\Phi}{\partial V_0}-\frac{\partial\phi}{\partial u_0}=0,$$
$$\mu_1\frac{\partial\Phi}{\partial U_0}+\mu_2\frac{\partial\Phi}{\partial V_0}-\frac{\partial\phi}{\partial v_0}=0,$$
$$\cdots\cdots\cdots\cdots$$

Likewise from $z'-z$

$$\lambda_1\frac{\partial\Psi}{\partial U_0}+\lambda_2\frac{\partial\Psi}{\partial V_0}-\frac{\partial\psi}{\partial u_0}=0,$$
$$\mu_1\frac{\partial\Psi}{\partial U_0}+\mu_2\frac{\partial\Psi}{\partial V_0}-\frac{\partial\psi}{\partial v_0}=0,$$
$$\cdots\cdots\cdots\cdots$$

Solving the corresponding equations in two of the sets we can determine the values of the λ's, μ's, ..., and <u>these values when</u>

substituted in the third set give the

$$\frac{n}{2}[n+3]$$

necessary and sufficient conditions for contact of order n.

The conditions for contact of order n are very greatly reduced if we let our surfaces be represented by the equations

$$S\begin{cases} x=f(u,v), \\ y=\phi(u,v), \\ z=\psi(u,v), \end{cases} \qquad S'\begin{cases} x=f(U,V), \\ y=\phi(U,V), \\ z=\Psi(U,V). \end{cases}$$

our common point corresponding to $U_0=u_0$, $V_0=v_0$.

In solving the corresponding equations of the groups determined by $x'-x$ and $y'-y$; we find

$$\lambda_1=1,\ \mu_1=0,=\alpha_1=\beta_1=\text{------}=0,$$

$$\mu_2=1,\ \lambda_2=0_2=\alpha_2=\beta_2=\text{------}=0.$$

Substituting these values of the coefficients in the set of equations for $z'-z$; we have

$$\frac{\partial \Psi}{\partial U_0}-\frac{\partial \psi}{\partial u_0}=0,$$

$$\frac{\partial \Psi}{\partial V_0}-\frac{\partial \psi}{\partial v_0}=0,$$

$$\frac{\partial^2 \Psi}{\partial U_0^2}-\frac{\partial^2 \psi}{\partial u_0^2}=0,$$

....

Therefore, the necessary and sufficient conditions for contact of order n at the point where $u_0=U_0$, $v_0=V_0$ is that the corresponding partial derivatives of the two z functions be equal up to and including derivatives of the nth order but that at least one pair of partial derivatives of (n+1)st order be not equal.

For the final consideration of contact let us denote the two surfaces by

$$z = \varphi(x,y) \quad , \quad z = \psi(x,y),$$

the common point corresponding to (x_0, y_0) . The two functions are equal for x_0, y_0 and furthermore $x' - x = 0$ and $y' - y = 0$. Thus the $\frac{n}{2}(n+3)$ equations for $z' - z$ must be satisfied. Expanding the two functions and determining the equations, we have the necessary and sufficient conditions for contact of order n to be that the corresponding partial derivatives must be equal up to and including derivatives of the nth order.

Examples.

Plane Curves.

Let us first consider the plan used to determine the equations for the C' curves of the first case where the equations of our curves are given by

$$C\begin{cases} x=f(t), \\ y=F(t), \end{cases} \qquad C'\begin{cases} x=\phi(u), \\ y=\Phi(u). \end{cases}$$

The equations of the C' curves of the following examples may be obtained in a similar manner.

Suppose we chose for the curve C the cubic $y=x^3$, and the common point as $(0,0)$. Let $x=t-1$, then $y=(t-1)^3$ and the common point corresponds to t = 1. Let C' be represented by $x=\phi(u)$; $y=\Phi(u)$; the common point corresponding to u = 0. Let us choose some arbitrary correspondence between the parameters and let us consider it as being given by

$$u-0=(t-1)+(t-1)^2.$$

We now have

$$C\begin{cases} x=t-1, \\ y=(t-1)^3, \end{cases} \qquad C'\begin{cases} x=\phi(u), \\ y=\Phi(u), \end{cases}$$

the common point $(0,0)$ corresponding to t = 1, u = 0, and the correspondence given by

$$u-0=(t-1)+(t-1)^2.$$

We wish to find some ϕ and Φ functions which will make close contact with C at the origin.

By Taylor's expansion,

$$x'=\phi(0)+(u-0)\,\phi'(0)+\frac{(u-0)^2}{\underline{|2}}\phi''(0)+\frac{(u-0)^3}{\underline{|3}}\phi'''(0)+\text{-----},$$

or
$$x' = \phi(0) + (t-1)\phi'(0) + (t-1)^2\phi'(0) + \frac{(t-1)^2}{2!}\phi''(0) + \frac{2(t-1)^3}{2!}\phi''(0) + \frac{(t-1)^4}{2!}\phi''(0) + \frac{(t-1)^3}{3!}\phi'''(0) + \frac{3(t-1)^4}{3!}\phi'''(0) + \cdots + \frac{(t-1)^4}{4!}\phi''''(0) + \cdots,$$

while $x = 0 + (t-1)$.

So for $x' - x$ to be an infinitesimal of, say, the fifth order with respect to t-1 we must have

$$\phi(0) = 0\ ,\ \phi'(0) = 1\ ,\ \phi''(0) = -2,\ \phi'''(0) = 12,\ \phi''''(0) = -120.$$

These conditions may be satisfied by selecting for $\phi(u)$ the expression

$$-5u^4 + 2u^3 - u^2 + u.$$

In the expansion for x' replace $\phi(0)$ ~~and~~ by $\Phi(0)$. This gives the expression for y'. For y of C we have

$$y = 0 + 0(t-1) + \frac{0(t-1)^2}{2!} + \frac{6(t-1)^3}{3!} + \frac{0(t-1)^4}{4!}.$$

The conditions upon $\Phi(u)$ may be satisfied by selecting for $\Phi(u)$ the expression

$$u^5 - 3u^4 + u^3.$$

Thus the curves given by the following equations:

$$C\begin{cases} x = (t-1), \\ y = (t-1)^3, \end{cases} \qquad C'\begin{cases} x = -5u^4 + 2u^3 - u^2 + u, \\ y = u^5 - 3u^4 + u^3, \end{cases}$$

have contact of the fourth order, since $x' - x$ and $y' - y$ are infinitesimals of the fifth order.

However, suppose we were given the equations of both curves

$$C\begin{cases} x = (t-1), \\ y = (t-1)^3, \end{cases} \qquad C'\begin{cases} x = -5u^4 + 2u^3 - u^2 + u, \\ y = u^5 - 3u^4 + u^3, \end{cases}$$

We know the shape of C and we wish to find out something about the curve C'. Since u is a common factor in x and y, the curve goes through the origin for $u = 0$. C goes through the origin for $t = 1$. Thus, the origin is common to both curves and corresponds to $t = 1$, $u = 0$.

Setting up a correspondence between <u>u</u> and <u>t</u>, we have,

(a) $$u - 0 = \lambda_1(t-1) + \lambda_2(t-1)^2 + \lambda_3(t-1)^3 + \lambda_4(t-1)^4 + \lambda_5(t-1)^5 + \cdots .$$

From C', we obtain

$$x' = 0 + (u-0) - (u-0)^2 + 2(u-0)^3 - 5(u-0)^4,$$

or, substituting from (a)

$$\begin{aligned} x' = 0 &+ \lambda_1(t-1) + \lambda_2(t-1)^2 + \lambda_3(t-1)^3 + \lambda_4(t-1)^4 + \lambda_5(t-1)^5 + \cdots \\ &- \lambda_1^2(t-1)^2 - 2\lambda_1\lambda_2(t-1)^3 - \lambda_2^2(t-1)^4 \\ &- 2\lambda_1\lambda_3(t-1)^4 - 2\lambda_1\lambda_4(t-1)^5 \\ &- 2\lambda_2\lambda_3(t-1)^5 - \cdots \\ &+ 2\lambda_1^3(t-1)^3 + 6\lambda_1^2\lambda_2(t-1)^4 + 6\lambda_1^2\lambda_3(t-1)^5 \\ &+ 6\lambda_1\lambda_2^2(t-1)^5 + \cdots \\ &- 5\lambda_1^4(t-1)^4 - 20\lambda_1^3\lambda_2(t-1)^5 - \cdots . \end{aligned}$$

From C, we have

$$x = 0 + (t-1).$$

Hence,

$$\lambda_1 = 1, \ \lambda_2 = 1, \ \lambda_3 = \lambda_4 = 0, \ \lambda_5 = +14, \ \cdots\cdots ,$$

or, $$u - 0 = (t-1) + (t-1)^2 + 14(t-1)^5 + \cdots\cdots .$$

In this manner we can determine values for the λ's such that $x' - x$ is of as high order as we wish. Now considering $y' - y$, we have

$$y' = (u-0)^3 - 3(u-0)^4 + (u-0)^5,$$

or

$$\begin{aligned} y' = (t-1)^3 &+ 3(t-1)^4 + 3(t-1)^5 + \cdots \\ &- 3(t-1)^4 - 12(t-1)^5 - \cdots \\ &+ (t-1)^5 + \cdots , \end{aligned}$$

and

$$y = (t-1)^3.$$

Hence $$y'-y=-8(t-1)^5+ \cdots\cdots ,$$
or, $y'-y$ is an infinitesimal of the fifth order.

Therefore, our curves have contact of the fourth order. This tells us that C and C' have five coincident points in common, that the two curves cross, and that C' resembles C very closely in the neighborhood of the origin.

The second case considered was the one in which the functions for x were the same. Let us suppose that we have our curves defined by the equations

$$C\begin{cases} x=2t, \\ y=t^2-2, \end{cases} \qquad C'\begin{cases} x=2u, \\ y=u^4+u^2-2. \end{cases}$$

The common point ($0,-2$) corresponds to $t_0=u_0=0$. Since $\lambda_1=1,\ \lambda_2=\lambda_3=\cdots=\lambda_n=0$, the correspondence between our parameters is given by

$$u-0=t-0+\lambda_{n+1}(t-0)^{n+1}+\cdots\cdots .$$

From the theoretical discussion of the case we found the necessary and sufficient condition for contact of order <u>n</u> to be that the functions for the y's must be equal for $t_0=u_0=0$ and that the corresponding first <u>n</u> derivatives must be equal. Considering our two curves, we have, after substituting zero for <u>u</u> and <u>t</u>,

for C , $\quad y=-2,\ y'=0,\ y''=2,\ y'''=0,\ y''''=0,$

and for C', $\quad y=-2,\ y'=0,\ y''=2,\ y'''=0,\ y''''=24.$

Hence the curves have contact of the third order, and the curve C' does not cross the parabolic curve C at (o, -2).

For the case in which our curves are represented by $\quad C\{y=F(x)\,,\quad C'\{y=\Phi(x),$

let us consider $\quad C\{y=-x^3,\quad C'\{y=x^4-2x^3.$

The curves have two common points $(0,0)$ and $(1,-1)$. Considering first the contact of the curves at the origin, we have, after substituting zero for x and y,

for C, $\quad y=0,\ y'=0,\ y''=0,\ y'''=-6,$

and for C', $\quad y=0\ ,\ y'=0\ ,\ y''=0,\ y'''=-12.$

So at $(0,0)$ our cufves have contact of second order, and C' crosses C.

At $(1,-1)$ after substituting unity for x and y, we have

for C, $\quad y=-1,\quad y'=-3,$

and for C', $\quad y=-1\ ,\ y'=-2.$

The curves at this point have contact of zero order, the curve C' simply crossing C.

For the final consideration of plane curves, let us investigate the contact of two curves concerning which we already have some information. For C, take the Witch of Agnesia, $x^2y=1-y$, and for C', the parabola, $y=1-x^2$. These curves have a point in common at (0,1) and, furthermore, we know that the curves do not cross.

Expressing C' parametrically, we have for our curves

$$C\{x^2y=1-y,\qquad C'\begin{cases} x=t+1, \\ y=-t^2-2t. \end{cases}$$

The common point $(0,1)$ corresponds to $t=-1$. Substituting the expressions for x and y of C' in C, we have

$$(t+1)^2(-t^2-2t)=1+t^2+2t,$$

or

$$t^4+4t^3+6t^2+4t+1=0.$$

For $t=-1$, this function and its first, second and third derivatives reduce to zero. But the fourth derivative has the value of 24. Therefore, the curves have contact of third order; they have four coincident points, and do not cross. This checks with what we expected and tells in addition that there are four coincident points where we might have thought there were only two.

Space Curves.

Let us take for our curve C, the twisted cubic and for C', a curve as indicated:

$$C\begin{cases} x=t, \\ y=t^2, \\ z=t^3, \end{cases} \qquad C'\begin{cases} x=4u^3-2u^2+u+1, \\ y=12u^3-3u^2+2u+1, \\ z=13u^3-3u^2+3u+1. \end{cases}$$

Our common point(1, 1, 1) corresponds to $t=1$, $u=0$. Let the correspondence be denoted by

$$u-0=\lambda_1(t-1)+\lambda_2(t-1)^2+\lambda_3(t-1)^3+\lambda_4(t-1)^4+\cdots\cdots.$$

From C', we have,

$$x'=1+(u-0)-2(u-0)^2+4(u-0)^3,$$

or

$$\begin{aligned} x'=1&+\lambda_1(t-1)+\lambda_2(t-1)^2+\lambda_3(t-1)^3+\lambda_4(t-1)^4+\lambda_5(t-1)^5+\cdots \\ &-2\lambda_1^2(t-1)^2-4\lambda_1\lambda_2(t-1)^3-2\lambda_2^2(t-1)^4 \\ &-4\lambda_1\lambda_3(t-1)^4-4\lambda_2\lambda_3(t-1)^5 \\ &-4\lambda_1\lambda_4(t-1)^5\cdots \\ &+4\lambda_1^3(t-1)^3+12\lambda_1^2\lambda_2(t-1)^4+12\lambda_1^2\lambda_3(t-1)^5 \\ &+12\lambda_1\lambda_2^2(t-1)^5+\cdots. \end{aligned}$$

From C,

$$x=1+(t-1).$$

Hence, $\lambda_1=1,\ \lambda_2=2,\ \lambda_3=4,\ \lambda_4=0,\ \lambda_5=64,\ \cdots\cdots.$

Thus, $u-0=(t-1)+2(t-1)^2+4(t-1)^3+64(t-1)^5+\cdots\cdots.$

Now considering $y'-y$, we have,

$$y'=1+2(u-0)-3(u-0)^2+12(u-0)^3,$$

or

$$\begin{aligned} y'=1+2(t-1)&+4(t-1)^2+8(t-1)^3 &&+128(t-1)^5+\cdots \\ &-3(t-1)^2-12(t-1)^3-36(t-1)^4 &&-48(t-1)^5\cdots\cdots \\ &+12(t-1)^3+72(t-1)^4 &&+288(t-1)^5+\cdots\cdots. \end{aligned}$$

But from C, $y=1+2(t-1)+(t-1)^2.$

Thus, $y'-y$ is an infinitesimal of the third order with respect to $t-1$.

Finally, considering the order of $z'-z$, we obtain,

$$z' = 1 + 3(u-0) - 3(u-0)^2 + 13(u-0)^3,$$

or

$$z' = 1 + 3(t-1) + 6(t-1)^2 + 12(t-1)^3 \qquad + 192(t-1)^5 + \cdots$$
$$-3(t-1)^2 - 12(t-1)^3 - 36(t-1)^4 \quad - 48(t-1)^5 - \cdots$$
$$+ 13(t-1)^3 + 78(t-1)^4 \quad + 312(t-1)^5 + \cdots,$$

and $$z = 1 + 3(t-1) + \frac{6(t-1)^2}{\underline{|2}} + \frac{6(t-1)^3}{\underline{|3}}.$$

So $z'-z$ is an infinitesimal of third order with respect to $t-1$.

Therefore, our curves have contact of second order and have three coincident points at (1, 1, 1).

For the simpler case in which the x's for both curves are represented by the same function of t and u respectively, let us consider the curves given by the following equations:

$$C\begin{cases} x = t, \\ y = t^2, \\ z = t^3, \end{cases} \qquad C'\begin{cases} x = u, \\ y = u^4 + u^2, \\ z = u^5 + u^3. \end{cases}$$

The common point (0, 0, 0) corresponds to $t = u = 0$. After substituting zero for t and u in the y functions and their derivatives, we have

for C, $\quad y = 0,\ y' = 0,\ y'' = 2,\ y''' = 0,\ y'''' = 0,$

and for C', $y = 0,\ y' = 0,\ y'' = 2,\ y''' = 0,\ y'''' = 24.$

Following the same procedure for the z functions, we have

for C, $\quad z = 0,\ z' = 0,\ z'' = 0,\ z''' = 6,\ z'''' = 0,\ z''''' = 0,$

and for C', $z = 0,\ z' = 0,\ z'' = 0,\ z''' = 6,\ z'''' = 0,\ z''''' = 120.$

Thus $y'-y$ is an infinitesimal of fourth order with respect to t and $z'-z$ is an infinitesimal of fifth order with respect to t. Hence, our curves have contact of the third order.

In our concluding case for space curves, let us consider for C the curve of intersection of a sphere, $x^2+y^2+z^2=4$, and a cylinder, $x^2+y^2=2y$. Or expressing it parametrically such that the x functions may be the same for C and C', we have curves given by the following equations:

$$C\begin{cases} x=\sqrt{2t-t^2}, \\ y=t, \\ z=\sqrt{4-2t}, \end{cases} \qquad C'\begin{cases} x=\sqrt{2t-t^2}, \\ y=t^5+t, \\ z=-\frac{5}{1024}t^4-\frac{3}{192}t^3-\frac{1}{16}t^2-\frac{1}{2}t+2. \end{cases}$$

The common point (0, 0, 2) corresponds to $t=0$. Taking the successive derivatives of the y functions, and substituting zero for t, we obtain,

for C, $y=0,\ y'=1,\ y''=0,\ y'''=0,\ y''''=0,\ y'''''=0,$

and for C', $y=0,\ y'=1,\ y''=0,\ y'''=0,\ y''''=0,\ y'''''=120.$

Following the same procedure for the z functions, we have

for C, $z=2,\ z'=-\frac{1}{2},\ z''=-\frac{1}{8},\ z'''=-\frac{3}{32},\ z''''=-\frac{15}{128},\ z'''''=-\frac{105}{512},$

and for C', $z=2,\ z'=-\frac{1}{2},\ z''=-\frac{1}{8},\ z'''=-\frac{3}{32},\ z''''=-\frac{15}{128},\ z'''''=0.$

Hence $y'-y$ and $z'-z$ are both infinitesimals of the fifth order with respect to t, and our curves have contact of fourth order.

Curve and Surface.

Suppose for our general case that our curve and surface are represented by the equations

$$C\begin{cases} x = t, \\ y = t^2 - 4, \\ z = 32t^3 - 126t^2 + 128t, \end{cases} \qquad S\begin{cases} x = u + v, \\ y = u - v, \\ z = 6u^2 - 4uv + 6v^2. \end{cases}$$

The common point (2, 0, 8) corresponds to $u = 1$, $v = 1$, $t = 2$.

Furthermore,

$$\frac{D(F, \Phi)}{D(u, v)} \neq 0 \quad \text{and} \quad \frac{D(\Phi, \psi)}{D(u, v)} \neq 0.$$

Let the relation between corresponding points be defined as follows:

$$u - 1 = \lambda_1(t-2) + \lambda_2(t-2)^2 + \lambda_3(t-2)^3 + \cdots,$$
$$v - 1 = \mu_1(t-2) + \mu_2(t-2)^2 + \mu_3(t-2)^3 + \cdots.$$

From $\overset{S}{\cancel{S}}$,

$$\begin{aligned} x' &= 2 + [(u-1) + (v-1)], \\ &= 2 + \lambda_1(t-2) + \lambda_2(t-2)^2 + \lambda_3(t-2)^3 + \lambda_4(t-2)^4 + \cdots, \\ &\quad + \mu_1(t-2) + \mu_2(t-2)^2 + \mu_3(t-2)^3 + \mu_4(t-2)^4 + \cdots. \end{aligned}$$

From C,

$$x = 2 + (t-2).$$

Thus,

(a)

$$\lambda_1 + \mu_1 = 1;\ \lambda_2 + \mu_2 = 0;\ \lambda_3 + \mu_3 = 0;\ \lambda_4 + \mu_4 = 0;\ \cdots.$$

But,

$$\begin{aligned} y' &= 0 + [(u-1) - (v-1)], \\ &= 0 + \lambda_1(t-2) + \lambda_2(t-2)^2 + \lambda_3(t-2)^3 + \lambda_4(t-2)^4 + \cdots \\ &\quad - \mu_1(t-2) - \mu_2(t-2)^2 - \mu_3(t-2)^3 - \mu_4(t-2)^4 - \cdots, \end{aligned}$$

while

$$y = 0 + 4(t-2) + \frac{2(t-2)^2}{\lfloor 2};$$

Hence,

(b)

$$\lambda_1 - \mu_1 = 4;\ \lambda_2 - \mu_2 = 1;\ \lambda_3 - \mu_3 = 0;\ \lambda_4 - \mu_4 = 0;\ \cdots.$$

Solving (a) and (b), we obtain:

$$\lambda_1 = \frac{5}{2};\ \lambda_2 = \frac{1}{2};\ \lambda_3 = \lambda_4 = \cdots = 0,$$

$$\mu_1 = -\frac{3}{2};\ \mu_2 = -\frac{1}{2};\ \mu_3 = \mu_4 = \cdots = 0.$$

Expanding the z functions and substituting the values just determined, we find,

$$z' = 8 + [8(u-1) + 8(v-1)] + \frac{1}{\underline{|2}}[12(u-1)^2 - 8(u-1)(v-1) + 12(v-1)^2],$$

or,

$$\begin{aligned} z' = {} & 8 + 20(t-2) + 4(t-2)^2 + 0 \quad + \quad 0 \quad + \cdots \\ & -12(t-2) - 4(t-2)^2 \\ & \qquad + \tfrac{75}{2}(t-2)^2 + 15(t-2)^3 + \tfrac{3}{2}(t-2)^4 + \cdots \\ & \qquad + 15(t-2)^2 + 8(t-2)^3 + (t-2)^4 + \cdots \\ & \qquad + \tfrac{27}{2}(t-2)^2 + 9(t-2)^3 + \tfrac{3}{2}(t-2)^4 + \cdots, \end{aligned}$$

and, $$z = 8 + 8(t-2) + \frac{132(t-2)^2}{\underline{|2}} + \frac{192(t-2)^3}{\underline{|3}}.$$

Hence, $z'-z$ is an infinitesimal of the fourth order, and our curve and surface have contact of the third order.

This means, then, that there is some curve on S passing throught the point (2, 0, 8) which makes contact of the third order with the curve C. By inspection of C it is easily seen that C is a curve on the parabolic cylinder $y = x^2 - 4$. It is easy to show that the curve on S that makes the third order contact with C is the curve of intersection of the parabolic cylinder and our surface.

As an illustration, of the second case, let us consider the following:

$$C\begin{cases} x = 2t^2, \\ y = 3t, \\ z = 24t^3 - \frac{63}{2}t^2 + 24t - 6, \end{cases} \qquad S\begin{cases} x = 2u^2, \\ y = 3v, \\ z = 6u^4 + \frac{9}{2}v^2. \end{cases}$$

The common point (2, 3, $10\frac{1}{2}$) corresponds to $t_0 = u_0 = v_0 = 1$. In this case

$$\lambda_1 = 1, \ \lambda_2 = \lambda_3 = \cdots = \lambda_n = 0,$$
$$\mu_1 = 1, \ \mu_2 = \mu_3 = \cdots = \mu_n = 0.$$

Substituting t for u and v of S we obtain $z=6t^4+\frac{9}{2}t^2$. Taking the derivatives of the z functions we have, after substituting unity for t,

for C, $z=10\frac{1}{2},\ z'=33,\ z''=81,\ z'''=144,\ z''''=0,$

and for S, $z=10\frac{1}{2},\ z'=33,\ z''=81,\ z'''=144,\ z''''=144.$

Hence the contact of C and S at $(2, 3, 10\frac{1}{2})$ is of the third order.

Finally, let us determine the order of contact between C and S when they are defined by means of the following equations:

$$C\begin{cases} x=t,\\ y=t^2,\\ z=t^4+\frac{3}{2}t^2, \end{cases} \qquad S\{2z=3x^2+y^2.$$

The common point (0, 0, 0) corresponds to $t_0=u_0=v_0=0$.

Substituting the functions for x, y, z, of C in S, we obtain

$$2t^4+3t^2-3t^2-t^4=0,$$

or

$$t^4=0.$$

This function, t^4, and its first three derivatives become zero for $t=0$, but the fourth derivative does not. Therefore, the curve C and the surface S have contact of the third order at the origin.

Order of Contact of Two Surfaces.

Let us take a paraboloid for our known surface in the three cases to be considered. By doing this the parametric expressions will be simplified.

For the first case, suppose our surfaces are defined by the following equations:

$$S\begin{cases} x = u+v, \\ y = u-v, \\ z = 25-2u^2-2v^2, \end{cases} \qquad S'\begin{cases} x = 2U-3V, \\ y = -U-3V, \\ z = -\frac{5}{3}U^3+18UV-18V^2+\frac{55}{3}. \end{cases}$$

The common point (1, 1, 23) corresponds to $u_0=0, v_0=0, U_0=2, V_0=1$.

Taking the partial derivatives of the functions with respect to their parameters and substituting those values in the equations given in the theoretical treatment for $x'-x$, we have,

$$\begin{aligned} 2\lambda_1 - 3\lambda_2 - 1 &= 0, \\ 2\mu_1 - 3\mu_2 - 1 &= 0, \\ 2\upsilon_1 - 3\upsilon_2 &= 0, \\ 2\alpha_1 - 3\alpha_2 &= 0, \\ 2\beta_1 - 3\beta_2 &= 0, \\ \ldots\ldots\ldots\ldots \end{aligned}$$

Substituting the values in the equations given for $y'-y$, we have

$$\begin{aligned} -\lambda_1 + 3\lambda_2 - 1 &= 0, \\ -\mu_1 + 3\mu_2 + 1 &= 0, \\ -\upsilon_1 + 3\upsilon_2 &= 0, \\ -\alpha_1 + 3\alpha_2 &= 0, \\ -\beta_1 + 3\beta_2 &= 0, \\ \ldots\ldots\ldots\ldots \end{aligned}$$

Solving corresponding equations of these two sets, we obtain,

$$\lambda_1 = 2,\ \mu_1 = 0,\ \upsilon_1 = 0,\ \alpha_1 = 0,\ \ldots\ldots,$$

$$\lambda_2 = 1,\ \mu_2 = -\frac{1}{3},\ \upsilon_2 = 0,\ \alpha_2 = 0,\ \ldots\ldots.$$

Considering next $z'-z$ and substituting the values just obtained, we have

$$\left.\begin{array}{r} 2(-2)+0+4\equiv 0 \\ -\frac{1}{3}(0)\ \equiv 0 \end{array}\right\},$$

$$\left.\begin{array}{r} -20+2(18)+\frac{1}{2}(-36)+2\equiv 0 \\ -\frac{2}{3}(18)-\frac{1}{3}(-36)\equiv 0 \\ \frac{1}{18}(-36)+2\equiv 0 \end{array}\right\},$$

but in the next set of four equations, the third derivatives are all zero for S while for S' the third partial with respect to u does not equal zero. So this set of four equations does not become zero and our contact is of the second order.

In the second case, let the x's and y's for both surfaces be represented by the same functions of different parameters, as:

$$S\begin{cases} x=u+v, \\ y=u-v, \\ z=25-2u^2-2v^2, \end{cases} \qquad S'\begin{cases} x=U+V, \\ y=U-V, \\ z=-U^3+U^2-3U-2UV^2+26. \end{cases}$$

The common point (1, 1, 23) corresponds to $u_0=U_0=1$, $v_0=V_0=0$.

Taking the partial derivatives of the two z functions and substituting unity for both u and U, and zero for v and V, we obtain

for S, $z=23,\ \frac{\partial z}{\partial u}=-4,\ \frac{\partial z}{\partial v}=0,\ \frac{\partial^2 z}{\partial u^2}=-4,\ \frac{\partial^2 z}{\partial u\partial v}=0,\ \frac{\partial^2 z}{\partial v^2}=-4,\ \frac{\partial^3 z}{\partial u^3}=0,$

for S', $z=23,\ \frac{\partial z}{\partial U}=-4,\ \frac{\partial z}{\partial V}=0,\ \frac{\partial^2 z}{\partial U^2}=-4,\ \frac{\partial^2 z}{\partial U\partial V}=0,\ \frac{\partial^2 z}{\partial V^2}=-4,\ \frac{\partial^3 z}{\partial U^3}=-6.$

Therefore, the curves have contact of second order. In other words every curve on S passing through (1, 1, 23) has contact of at least second order with some curve on S' which passes through this same point.

Finally let the surfaces be defined in the following manner: $S\{z = x^2+y^2\,, \quad S'\{z = x^3y+x^2+y^2+y^4.$

The common point (0, 0, 0) corresponds to $x_0 = y_0 = 0$. Taking the partial derivatives of the two functions and substituting zero for x and y, we have

for S, $z=0,\ \frac{\partial z}{\partial x}=0,\ \frac{\partial z}{\partial y}=0,\ \frac{\partial^2 z}{\partial x^2}=2,\ \frac{\partial^2 z}{\partial x\partial y}=0,\ \frac{\partial^2 z}{\partial y^2}=2,$

$$\frac{\partial^3 z}{\partial x^3}=\frac{\partial^3 z}{\partial x^2\partial y}=\frac{\partial^3 z}{\partial x\partial y^2}=\frac{\partial^3 z}{\partial y^3}=\frac{\partial^4 z}{\partial x^4}=\text{------}=\frac{\partial^4 z}{\partial y^4}=0,$$

and for S', $z=0,\ \frac{\partial z}{\partial x}=0,\ \frac{\partial z}{\partial y}=0,\ \frac{\partial^2 z}{\partial x^2}=2,\ \frac{\partial^2 z}{\partial x\partial y}=0,\ \frac{\partial^2 z}{\partial y^2}=2,$

$$\frac{\partial^3 z}{\partial x^3}=\frac{\partial^3 z}{\partial x^2\partial y}=\frac{\partial^3 z}{\partial x\partial y^2}=\frac{\partial^3 z}{\partial y^3}=\frac{\partial^4 z}{\partial x^4}=\text{------}=0,\ \frac{\partial^4 z}{\partial y^4}=24.$$

Therefore, our surfaces have contact of the third order.